A lecture on
Matrix Formulation of Angular Momentum

Paper- 201
M.Sc. 2 sem

Matrix Formulation of Angular Momentum

Let us now introduce a more general angular momentum operator $\hat{\vec{J}}$ that is defined by its three components $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$, which satisfy the following commutation relations:

$$\boxed{[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z, \qquad [\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x, \qquad [\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y,}$$

or equivalently by

$$\hat{\vec{J}} \times \hat{\vec{J}} = i\hbar\hat{\vec{J}}.$$

Since $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$ do not mutually commute, they cannot be simultaneously diagonalized; that is, they do not possess common eigenstates. The square of the angular momentum,

$$\hat{\vec{J}}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2,$$

is a scalar operator; hence it commutes with $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$:

$$[\hat{\vec{J}}^2, \hat{J}_k] = 0,$$

where k stands for x, y, and z. For instance, in the the case $k = x$ we have

$$\begin{aligned}
[\hat{\vec{J}}^2, \hat{J}_x] &= [\hat{J}_x^2, \hat{J}_x] + \hat{J}_y[\hat{J}_y, \hat{J}_x] + [\hat{J}_y, \hat{J}_x]\hat{J}_y + \hat{J}_z[\hat{J}_z, \hat{J}_x] + [\hat{J}_z, \hat{J}_x]\hat{J}_z \\
&= \hat{J}_y(-i\hbar\hat{J}_z) + (-i\hbar\hat{J}_z)\hat{J}_y + \hat{J}_z(i\hbar\hat{J}_y) + (i\hbar\hat{J}_y)\hat{J}_z \\
&= 0,
\end{aligned}$$

because $[\hat{J}_x^2, \hat{J}_x] = 0$, $[\hat{J}_y, \hat{J}_x] = -i\hbar\hat{J}_z$, and $[\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y$. We should note that the operators $\hat{J}_x$, $\hat{J}_y$, $\hat{J}_z$, and $\hat{\vec{J}}^2$ are all Hermitian; their eigenvalues are real.

Eigenstates and Eigenvalues of the angular Momentum Operator

Since $\vec{J}^2$ commutes with $\hat{J}_x$, $\hat{J}_y$ and $\hat{J}_z$, each component of $\vec{J}$ can be separately diagonalized (hence it has simultaneous eigenfunctions) with $\hat{\vec{J}}^2$. But since the components $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$ do not mutually commute, we can choose only one of them to be simultaneously diagonalized with $\hat{\vec{J}}^2$. By convention we choose $\hat{J}_z$. There is nothing special about the z-direction; we can just as well take $\hat{\vec{J}}^2$ and $\hat{J}_x$ or $\hat{\vec{J}}^2$ and $\hat{J}_y$.

Let us now look for the joint eigenstates of $\hat{\vec{J}}^2$ and $\hat{J}_z$ and their corresponding eigenvalues. Denoting the joint eigenstates by $|\alpha, \beta\rangle$ and the eigenvalues of $\hat{\vec{J}}^2$ and $\hat{J}_z$ by $\hbar^2\alpha$ and $\hbar\beta$, respectively, we have

$$\hat{\vec{J}}^2 \,|\, \alpha, \beta\rangle = \hbar^2\alpha \,|\, \alpha, \beta\rangle,$$
$$\hat{J}_z \,|\, \alpha, \beta\rangle = \hbar\beta \,|\, \alpha, \beta\rangle. \qquad (5.25)$$

The factor $\hbar$ is introduced so that α and β are dimensionless; recall that the angular momentum has the dimensions of $\hbar$ and that the physical dimensions of $\hbar$ are: $[\hbar]$ = energy $\times$ time. For simplicity, we will assume that these eigenstates are orthonormal:

$$\langle \alpha', \beta' \mid \alpha, \beta \rangle = \delta_{\alpha', \alpha}\delta_{\beta', \beta}.$$

Now we need to introduce *raising* and *lowering* operators $\hat{J}_+$ and $\hat{J}_-$, just as we did when we studied the harmonic oscillator in Chapter 4:

$$\boxed{\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y.}$$

This leads to

$$\hat{J}_x = \frac{1}{2}(\hat{J}_+ + \hat{J}_-), \qquad \hat{J}_y = \frac{1}{2i}(\hat{J}_+ - \hat{J}_-);$$

hence

$$\hat{J}_x^2 = \frac{1}{4}(\hat{J}_+^2 + \hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+ + \hat{J}_-^2), \qquad \hat{J}_y^2 = -\frac{1}{4}(\hat{J}_+^2 - \hat{J}_+\hat{J}_- - \hat{J}_-\hat{J}_+ + \hat{J}_-^2).$$

we can easily obtain the following commutation relations:

$$[\hat{\vec{J}}^2, \hat{J}_\pm] = 0, \qquad [\hat{J}_+, \hat{J}_-] = 2\hbar\hat{J}_z, \qquad [\hat{J}_z, \hat{J}_\pm] = \pm\hbar\hat{J}_\pm.$$

In addition, $\hat{J}_+$ and $\hat{J}_-$ satisfy

$$\hat{J}_+\hat{J}_- = \hat{J}_x^2 + \hat{J}_y^2 + \hbar\hat{J}_z = \hat{\vec{J}}^2 - \hat{J}_z^2 + \hbar\hat{J}_z,$$

$$\hat{J}_-\hat{J}_+ = \hat{J}_x^2 + \hat{J}_y^2 - \hbar\hat{J}_z = \hat{\vec{J}}^2 - \hat{J}_z^2 - \hbar\hat{J}_z.$$

These relations lead to

$$\boxed{\hat{\vec{J}}^2 = \hat{J}_\pm\hat{J}_\mp + \hat{J}_z^2 \mp \hbar\hat{J}_z,}$$

which in turn yield

$$\boxed{\hat{\vec{J}}^2 = \frac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+) + \hat{J}_z^2.}$$

Let us see how $\hat{J}_\pm$ operate on $| \alpha, \beta\rangle$. First, since $\hat{J}_\pm$ do not commute with $\hat{J}_z$, the kets $| \alpha, \beta\rangle$ are not eigenstates of $\hat{J}_\pm$. (

$$\hat{J}_z(\hat{J}_\pm | \alpha, \beta\rangle) = (\hat{J}_\pm\hat{J}_z \pm \hbar\hat{J}_\pm) | \alpha, \beta\rangle = \hbar(\beta \pm 1)(\hat{J}_\pm | \alpha, \beta\rangle);$$

hence the ket $(\hat{J}_\pm \mid \alpha,\ \beta\rangle)$ is an eigenstate of $\hat{J}_z$ with eigenvalues $\hbar(\beta \pm 1)$. Now since $\hat{J}_z$ and $\hat{\vec{J}}^2$ commute, $(\hat{J}_\pm \mid \alpha,\ \beta\rangle)$ must also be an eigenstate of $\hat{\vec{J}}^2$. The eigenvalue of $\hat{\vec{J}}^2$ when acting on $\hat{J}_\pm \mid \alpha,\ \beta\rangle$ can be determined by making use of the commutator $[\hat{\vec{J}}^2, \hat{J}_\pm] = 0$. The state $(\hat{J}_\pm \mid \alpha,\ \beta\rangle)$ is also an eigenstate of $\hat{\vec{J}}^2$ with eigenvalue $\hbar^2\alpha$:

$$\hat{\vec{J}}^2(\hat{J}_\pm \mid \alpha,\ \beta\rangle) = \hat{J}_\pm \hat{\vec{J}}^2 \mid \alpha,\ \beta\rangle = \hbar^2\alpha(\hat{J}_\pm \mid \alpha,\ \beta\rangle).$$

From (5.33) and (5.34) we infer that when $\hat{J}_\pm$ acts on $\mid \alpha,\ \beta\rangle$, it does not affect the first quantum number α, but it raises or lowers the second quantum number β by one unit. That is, $J_\pm \mid \alpha,\ \beta\rangle$ is proportional to $\mid \alpha,\ \beta \pm 1\rangle$:

$$\hat{J}_\pm \mid \alpha,\ \beta\rangle = C^\pm_{\alpha\beta} \mid \alpha,\ \beta \pm 1\rangle.$$

We will determine the constant $C^\pm_{\alpha\beta}$ later on.

Note that, for a given eigenvalue α of $\hat{\vec{J}}^2$, there exists an *upper limit* for the quantum number β. This is due to the fact that the operator $\hat{\vec{J}}^2 - \hat{J}_z^2$ is positive, since the matrix elements of $\hat{\vec{J}}^2 - \hat{J}_z^2 = \hat{J}_x^2 + \hat{J}_y^2$ are ≥ 0; we can therefore write

$$\langle \alpha, \beta \mid \hat{\vec{J}}^2 - \hat{J}_z^2 \mid \alpha, \beta \rangle = \hbar^2(\alpha - \beta^2) \geq 0, \quad \Longrightarrow \quad \alpha \geq \beta^2.$$

Since β has an upper limit β_{max}, there must exist a state $\mid \alpha, \beta_{max} \rangle$ which cannot be raised further:

$$\hat{J}_+ \mid \alpha, \beta_{max} \rangle = 0.$$

Using this relation along with $\hat{J}_-\hat{J}_+ = \hat{\vec{J}}^2 - \hat{J}_z^2 - \hbar \hat{J}_z$, we see that $\hat{J}_-\hat{J}_+ \mid \alpha, \beta_{max} \rangle = 0$ or

$$(\hat{\vec{J}}^2 - \hat{J}_z^2 - \hbar \hat{J}_z) \mid \alpha, \beta_{max} \rangle = \hbar^2(\alpha - \beta_{max}^2 - \beta_{max}) \mid \alpha, \beta_{max} \rangle;$$

hence

$$\alpha = \beta_{max}(\beta_{max} + 1).$$

After n successive applications of $\hat{J}_-$ on $|\,\alpha,\ \beta_{max}\rangle$, we must be able to reach a state $|\,\alpha,\ \beta_{min}\rangle$ which cannot be lowered further:

$$\hat{J}_-\,|\,\alpha,\ \beta_{min}\rangle = 0.$$

Using $\hat{J}_+\hat{J}_- = \hat{\vec{J}}^2 - \hat{J}_z^2 + \hbar\hat{J}_z$, and l

$$\alpha = \beta_{min}(\beta_{min} - 1).$$

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$$\beta_{max} = -\beta_{min}.$$

Since β_{min} was reached by n applications of $\hat{J}_-$ on $|\,\alpha,\ \beta_{max}\rangle$, it follows that

$$\beta_{max} = \beta_{min} + n,$$

and since $\beta_{min} = -\beta_{max}$ we conclude that

$$\beta_{max} = \frac{n}{2}.$$

Hence β_{max} can be integer or half-odd-integer, depending on n being even or odd.

It is now appropriate to introduce the notation j and m to denote β_{max} and β, respectively:

$$j = \beta_{max} = \frac{n}{2}, \qquad m = \beta; \tag{5.45}$$

hence the eigenvalue of $\hat{\vec{J}}^2$ is given by

$$\alpha = j(j+1).$$

Now since $\beta_{min} = -\beta_{max}$, and with n positive, we infer that the allowed val
between $-j$ and $+j$:

$$\boxed{-j \leq m \leq j.} \tag{5.47}$$

The results obtained thus far can be summarized as follows: the eigenvalues of $\hat{\vec{J}}^2$ a
corresponding to the joint eigenvectors $|\, j,\ m\rangle$ are given, respectively, by $\hbar^2 j(j+1)$ and $\hbar m$:

$$\boxed{\hat{\vec{J}}^2 \,|\, j,\ m\rangle = \hbar^2 j(j+1) \,|\, j,\ m\rangle \quad \text{and} \quad \hat{J}_z \,|\, j,\ m\rangle = \hbar m \,|\, j,\ m\rangle,} \tag{5.48}$$

where $j = 0, 1/2, 1, 3/2, \ldots$ and $m = -j, -(j-1), \ldots, j-1, j$. So for each j there are $2j+1$ values of m. For example, if $j = 1$ then m takes the three values $-1, 0, 1$; and if $j = 5/2$ then m takes the six values $-5/2, -3/2, -1/2, 1/2, 3/2, 5/2$. The values of j are either integer or half-integer. We see that the spectra of the angular momentum operators $\hat{J}^2$ and $\hat{J}_z$ are discrete. Since the eigenstates corresponding to different angular momenta are orthogonal, and since the angular momentum spectra are discrete, the orthonormality condition is

$$\boxed{\langle j', m' \mid j, m \rangle = \delta_{j',j}\delta_{m',m}.}$$

Let us now determine the eigenvalues of $\hat{J}_{\pm}$ within the $\{| j, m\rangle\}$ basis; $| j, m\rangle$ is not an eigenstate of $\hat{J}_{\pm}$. We can rewrite equation (5.35) as

$$\hat{J}_{\pm} | j, m\rangle = C^{\pm}_{jm} | j, m \pm 1\rangle.$$

We are going to derive C^{+}_{jm} and then infer C^{-}_{jm}. Since $| j, m\rangle$ is normalized, we can use to obtain the following two expressions:

$$\begin{aligned}(\hat{J}_{+} | j, m\rangle)^{\dagger} (\hat{J}_{+} | j, m\rangle) &= |C^{+}_{jm}|^2 \langle j, m+1 | j, m+1\rangle = |C^{+}_{jm}|^2, \\ \left|C^{+}_{jm}\right|^2 &= \langle j, m|\hat{J}_{-}\hat{J}_{+} | j, m\rangle.\end{aligned}$$

But since $\hat{J}_{-}\hat{J}_{+}$ is equal to $(\hat{\vec{J}}^2 - \hat{J}_z^2 - \hbar\hat{J}_z)$, and assuming the arbitrary phase of C^{+}_{jm} to be zero, we conclude that

$$C^{+}_{jm} = \sqrt{\langle j, m|\hat{\vec{J}}^2 - \hat{J}_z^2 - \hbar\hat{J}_z | j, m\rangle} = \hbar\sqrt{j(j+1) - m(m+1)}.$$

By analogy with C^{+}_{jm} we can easily infer the expression for C^{-}_{jm}:

$$C^{-}_{jm} = \hbar\sqrt{j(j+1) - m(m-1)}.$$

Thus, the eigenvalue equations for $\hat{J}_+$ and $\hat{J}_-$ are given by

$$\boxed{\hat{J}_\pm \mid j,\, m\rangle = \hbar\sqrt{j(j+1) - m(m \pm 1)}\ \mid j,\, m \pm 1\rangle}$$

or

$$\boxed{\hat{J}_\pm \mid j,\, m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}\ \mid j,\, m \pm 1\rangle,}$$

which in turn leads to the two relations:

$$\begin{aligned}\hat{J}_x \mid j,\, m\rangle &= \frac{1}{2}(\hat{J}_+ + \hat{J}_-) \mid j,\, m\rangle \\ &= \frac{\hbar}{2}\left[\sqrt{(j-m)(j+m+1)}\ \mid j,\, m+1\rangle + \sqrt{(j+m)(j-m+1)}\ \mid j,\, m-1\rangle\right],\end{aligned}$$

$$\hat{J}_y \mid j,\ m\rangle = \frac{1}{2i}(\hat{J}_+ - \hat{J}_-) \mid j,\ m\rangle$$
$$= \frac{\hbar}{2i}\left[\sqrt{(j-m)(j+m+1)} \mid j,\ m+1\rangle - \sqrt{(j+m)(j-m+1)} \mid j,\ m-1\rangle\right]$$

The expectation values of $\hat{J}_x$ and $\hat{J}_y$ are therefore zero:

$$\langle j,\ m \mid \hat{J}_x \mid j,\ m\rangle = \langle j,\ m \mid \hat{J}_y \mid j,\ m\rangle = 0$$

We will show later in (5.208) that the expectation values $\langle j,\ m \mid \hat{J}_x^2 \mid j,\ m\rangle$ and $\langle j,\ m \mid \hat{J}_y^2 \mid j,\ m\rangle$ are equal and given by

$$\boxed{\langle \hat{J}_x^2\rangle = \langle \hat{J}_y^2\rangle = \frac{1}{2}\left[\langle j,\ m \mid \hat{\vec{J}}^2 \mid j,\ m\rangle - \langle j,\ m \mid \hat{J}_z^2 \mid j,\ m\rangle\right] = \frac{\hbar^2}{2}\left[j(j+1) - m^2\right].}$$

Matrix Representation of Angular Momentum

The formalism of the previous section is general and independent of any particular representation. There are many ways to represent the angular momentum operators and their eigenstates. In this section we are going to discuss the matrix representation of angular momentum where eigenkets and operators will be represented by column vectors and square matrices, respectively. This is achieved by expanding states and operators in a discrete basis. We will see later how to represent the orbital angular momentum in the position representation.

Since $\hat{\vec{J}}^2$ and $\hat{J}_z$ commute, the set of their common eigenstates $\{| j, m\rangle\}$ can be chosen as a basis; this basis is discrete, orthonormal, and complete. For a given value of j, the orthonormalization condition for this base is given by , and the completeness condition is expressed by

$$\sum_{m=-j}^{+j} | j, m\rangle\langle j, m | = \hat{I},$$

where $\hat{I}$ is the unit matrix. The operators $\hat{\vec{J}}^2$ and $\hat{J}_z$ are diagonal in the basis given by their eigenstates

$$\langle j', m' | \hat{\vec{J}}^2 | j, m\rangle = \hbar^2 j(j+1)\delta_{j',j}\delta_{m',m},$$

$$\langle j', m' | \hat{J}_z | j, m\rangle = \hbar m\delta_{j',j}\delta_{m',m}.$$

Thus, the matrices representing $\hat{\vec{J}}^2$ and $\hat{J}_z$ in the $\{| j, m\rangle\}$ eigenbasis are diagonal, their diagonal elements being equal to $\hbar^2 j(j+1)$ and $\hbar m$, respectively.

Now since the operators $\hat{J}_\pm$ do not commute with $\hat{J}_z$, they are represented in the $\{| j, m\rangle\}$ basis by matrices that are not diagonal:

$$\boxed{\langle j', m' \mid \hat{J}_\pm \mid j, m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\, \delta_{j',j}\delta_{m',m\pm 1}.}$$

We can infer the matrices of $\hat{J}_x$ and $\hat{J}_y$ from (5.57) and (5.58):

$$\begin{aligned}
\langle j', m' \mid \hat{J}_x \mid j, m\rangle &= \frac{\hbar}{2}\Big[\sqrt{j(j+1) - m(m+1)}\delta_{m', m+1} \\
&\quad + \sqrt{j(j+1) - m(m-1)}\delta_{m', m-1}\Big]\delta_{j', j}, \\
\langle j', m' | \hat{J}_y \mid j, m\rangle &= \frac{\hbar}{2i}\Big[\sqrt{j(j+1) - m(m+1)}\delta_{m', m+1} \\
&\quad - \sqrt{j(j+1) - m(m-1)}\delta_{m', m-1}\Big]\delta_{j', j}.
\end{aligned}$$

For $j = 1$ the allowed values of m are $-1, 0, 1$. The joint eigenstates of $\hat{\vec{J}}^2$ and $\hat{J}_z$ are $|1, -1\rangle$, $|1, 0\rangle$, and $|1, 1\rangle$. The matrix representations of the operators $\hat{\vec{J}}^2$ and $\hat{J}_z$

$$\hat{\vec{J}}^2 = \begin{pmatrix} \langle 1, 1 | \hat{\vec{J}}^2 | 1, 1\rangle & \langle 1, 1 | \hat{\vec{J}}^2 | 1, 0\rangle & \langle 1, 1 | \hat{\vec{J}}^2 | 1, -1\rangle \\ \langle 1, 0 | \hat{\vec{J}}^2 | 1, 1\rangle & \langle 1, 0 | \hat{\vec{J}}^2 | 1, 0\rangle & \langle 1, 0 | \hat{\vec{J}}^2 | 1, -1\rangle \\ \langle 1, -1 | \hat{\vec{J}}^2 | 1, 1\rangle & \langle 1, -1 | \hat{\vec{J}}^2 | 1, 0\rangle & \langle 1, -1 | \hat{\vec{J}}^2 | 1, -1\rangle \end{pmatrix}$$

$$J^2 = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{J}_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

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